Second-order optimization of frames with nonlinear behavior

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Abstract

In this paper an integrated formulation for the structural optimization of frames with nonlinear material behavior is presented. Member sizes, generalized displacements and internal forces are treated as variables of the nonlinear program. The objective function is the cost of the structure. Equilibrium equations and plastic hinge complementary conditions are treated as equality constraints. Stresses, displacements and plastic hinge rotations may be limited. Plastic hinges are automatically located in the most critical position. The nonlinear program is solved by a second-order Lagrange-Newton algorithm. The computer implementation parses all the expressions and calculates first and second derivatives of the Lagrangian. The sparsity of the Hessian matrix is taken into account and several numerical techniques are used to improve the efficiency and the reliability of the optimization process. A numerical example is presented.

1 Introduction

The traditional formulation for the optimization of structures with nonlinear behavior assumes that the material has rigid-plastic or elastic-plastic behavior and imposes no limit on the plastic deformations [6] [9] [10]. In these cases only the ultimate state or collapse mechanism is considered. In a more recent approach, sensitivity analysis information is evaluated and an optimizer is used [1]. When the material behavior or its derivatives are not continuous, sensitivity of the structural response to changes in the design variables may be erratic, causing instabilities in the iteration process.

In most design problems the behavior of the structure should be considered nonlinear, but the ultimate mechanism cannot be attained due to limited plastic deformation of the constituent materials (e.g., reinforced concrete). The formulation presented in this paper allows for the optimal design of frames with limited plastic hinge rotations. In this case, the ultimate state corresponds to an intermediate situation that is located between the linear elastic behavior and the collapse mechanism. This approach leads to a more acceptable nonlinear design criterion, when the constituent materials are unable to exhibit a constant or growing stress under large deformations. The combination of the referred formulation with an accurate and efficient second-order optimization algorithm constitutes a useful tool for the design of frame type structures. This optimization algorithm is based on the Lagrange-Newton method and is described below.

2 Nonlinear programming

The nonlinear program for the minimization of a function subjected to inequality and equality constraints may adopt the following general form

$$\textit{Minimize} \quad f\,(\underset{\sim}{x}) \qquad \underset{\sim}{x} = \left(x_1, \ldots, x_n\right) \tag{1}$$

$$\textit{subjected to}$$

$$\underset{\sim}{g}\,(\underset{\sim}{x}) \le \underset{\sim}{0} \qquad \underset{\sim}{g} = \left(g_1, \ldots, g_m\right) \tag{2}$$

$$\underset{\sim}{h}\,(\underset{\sim}{x}) = \underset{\sim}{0} \qquad \underset{\sim}{h} = \left(h_1, \ldots, h_p\right) \tag{3}$$

Squared slack variables may be added to the inequality constraints, leading to a generic treatment of all the constraints as equalities [2].

$$g_i(\underset{\sim}{x}) \le 0 \quad \rightarrow \quad g_i(\underset{\sim}{x}) + s_i^2 = 0 \qquad \left(i = 1, \ldots, m\right) \tag{4}$$

The Lagrangian of the nonlinear program (1-3) with the substitution (4) is given by [8]

$$L(\underset{\sim}{X}) = f\,(\underset{\sim}{x}) + \sum_{k=1}^{m} \lambda_k^g \left[g_k(\underset{\sim}{x}) + s_k^2 \right] + \sum_{k=1}^{p} \lambda_k^h h_k(\underset{\sim}{x}) \tag{5}$$

In equation (5), $\underset{\sim}{\lambda}^g$ and $\underset{\sim}{\lambda}^h$ are the Lagrange multipliers associated with the inequality and equality constraints respectively and $\underset{\sim}{X}$ is a vector that groups all the variables ($\underset{\sim}{x}$, $\underset{\sim}{s}$, $\underset{\sim}{\lambda}^g$ and $\underset{\sim}{\lambda}^h$). The solution of the nonlinear program (1)-(3) is also a solution of the following system of nonlinear equations [7]

$$\nabla L(\underset{\sim}{X}) = \underset{\sim}{0} \tag{6}$$

When the Newton method is used to solve (6), the optimization method is termed Lagrange-Newton and, for an initial solution sufficiently close to a stationary point, its convergence rate is quadratic. For each Newton iteration (q) the following system of linear equations must be solved

$$\underset{\sim}{H}(\underset{\sim}{X}^{q-1})\ \Delta\underset{\sim}{X}^{q}+\nabla L(\underset{\sim}{X}^{q-1})=\underset{\sim}{0} \tag{7}$$

The Hessian of the Lagrangian $(\underset{\sim}{H})$ is a symmetric matrix and in most cases is very sparse. The vector $\Delta\underset{\sim}{X}^{q}$ is multiplied by a line search parameter and is used to update the current solution [12].

3 NLP software

The robustness and the efficiency of the Lagrange-Newton method depend on a set of numerical techniques that were implemented in a computer code called NEWTOP [3]. With this program, structural optimization problems with over 4000 independent design variables and 20000 constraints have been successfully solved [4]. Some of the aforementioned techniques are briefly described in this section.

To facilitate the symbolic manipulation of the functions that define the mathematical program (f, g_i and h_i), their type is restricted to the following generalized polynomials

$$\sum_{t=1}^{T} c_t \prod_{i=1}^{n} x_i^{a_{ti}} \tag{8}$$

The following example clarifies the type of expressions defined by (8)

$$f\,(\underset{\sim}{x})=5.9x_1^2x_4^{-3}-3.1x_2+2.7x_1^{-1}x_3x_5^2-1.8 \tag{9}$$

Derivation and evaluation of this type of expressions are straightforward and can be efficiently performed by the computer. First and second derivatives of each function in order to all the variables can thus be accurately calculated.

In most optimization problems the coefficient matrix of the system of linear equations (7) is very sparse. This property of the Hessian matrix leads to huge savings in storage space and allows for the development of efficient solution algorithms. Gaussian elimination adapted to the sparsity pattern of the Hessian and preconditioned conjugate gradients have been successfully implemented.

The robustness of the iteration process can be significantly improved with the application of some scaling techniques. In the program NEWTOP the variables $\underset{\sim}{x}$, the slack variables $\underset{\sim}{s}$ and the Lagrange multipliers $\underset{\sim}{\lambda}^g$ and $\underset{\sim}{\lambda}^h$ are linearly scaled. The scaling coefficients are calculated before the first iteration and depend on the initial solution. When the initial and the optimal solutions are very distant from each other, the iterative process may need to be restarted several times.

NEWTOP input data is a text file containing a readable nonlinear program that can be prepared by the user or generated by another program. The former strategy is only suitable for small size problems, while the later requires the development of a specific computer program for each type of optimization problem. NEWTOP input file is parsed, validated and stored in a data structure that is more suitable to be manipulated during the iteration process. One of the most useful techniques implemented in this phase is the automatic substitution of elementary constraints of the type $x_i = c$ or $x_i = c\,x_j$. After the substitution, the remaining expressions become less complex and the number of variables and constraints is reduced. This feature simplifies significantly the task of generating a mathematical program for each new type of problem.

The solution of a wide variety of structural optimization problems indicates that the NEWTOP program is reliable, accurate and reasonably efficient in modern computational platforms.

4 Frames with nonlinear material behavior

For the type of structural optimization problems presented in this paper, the material of each frame bar is assumed to have elastic-plastic behavior. The elastic phase is linear and, in the plastic phase, deformations are allowed to increase under constant stress. Depending on the structural material, deformations cannot grow indefinitely, due to brittle behavior or extensive damage. When the behavior of the structure is dominated by bending and the material behavior is supposed to be linear-perfectly plastic, the moment-curvature diagram can be simplified as shown in Figure 1 [6]. The expression of the fully plastic moment (M_p) depends on the shape of the cross section. The value $k_{\max}$ is also problem dependent and must be specified before the beginning of the optimization process.

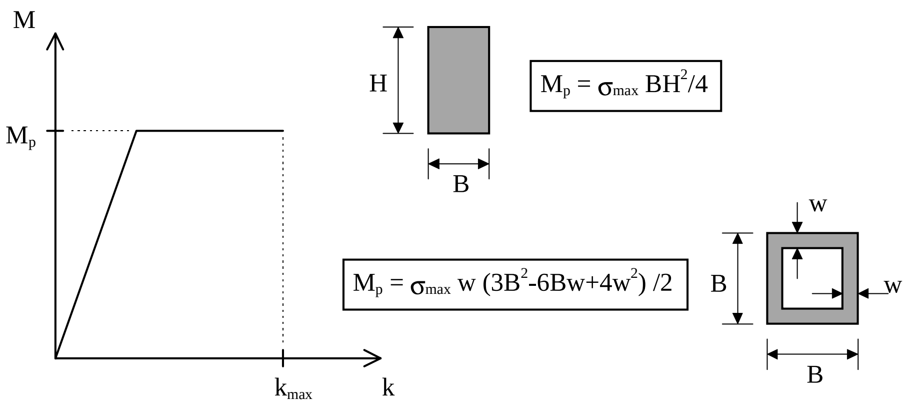

Figure 1: Simplified moment-curvature diagram.

When a frame bar is subjected to a distributed load (Figure 2), three plastic hinges (*A*, *B* and *C*) may potentially be formed.

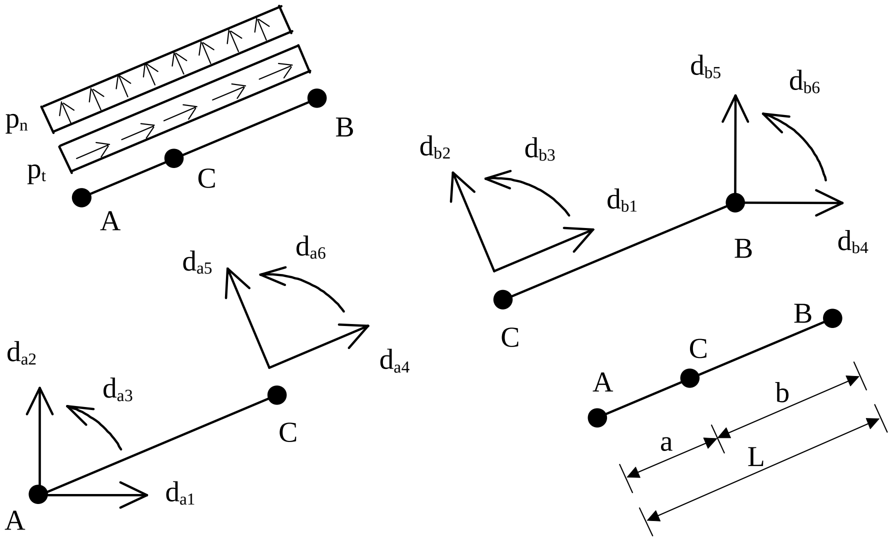

Figure 2: Plastic hinges and degrees of freedom of a beam.

The length of the beam (L) is known, but the position of hinge C is to be determined by the optimization algorithm. With this strategy there is no need to predefine a set of candidate points to be later selected as hinge C [11]. Each beam of length L is thus subdivided in two bars of length a and b respectively, whose degrees of freedom $\underset{\sim}{d}_a$ and $\underset{\sim}{d}_b$ are represented in Figure 2.

The degrees of freedom of nodes A and B are in the global coordinate system, while those of node C are in a local coordinate system, whose first axis is parallel to the beam. The stiffness matrices $\underset{\sim}{K}_a$ and $\underset{\sim}{K}_b$ of the beams AC and CB are defined in these coordinate systems. In the following equilibrium equations, $\underset{\sim}{F}$ are forces at the bar extremities in correspondence to $\underset{\sim}{d}$. Vector $\underset{\sim}{P}$ represents nodal forces equivalent to the distributed loads p_t and p_n.

$$\underset{\sim}{K}_a \, \underset{\sim}{d}_a = \underset{\sim}{F}_a + \underset{\sim}{P}_a \tag{10}$$

$$\underset{\sim}{K}_b \, \underset{\sim}{d}_b = \underset{\sim}{F}_b + \underset{\sim}{P}_b \tag{11}$$

Each frame node (e.g., node N in Figure 3) connects several beams with forces $\underset{\sim}{F}_a$ or $\underset{\sim}{F}_b$ in their extremities. The node itself may be loaded with external forces $\underset{\sim}{Q}$ and/or reactions $\underset{\sim}{R}$ and the following equilibrium equations can be written

$$\underset{\sim}{F}_a{}' + \cdots + \underset{\sim}{F}_b{}'' + \cdots = \underset{\sim}{Q} + \underset{\sim}{R} \tag{12}$$

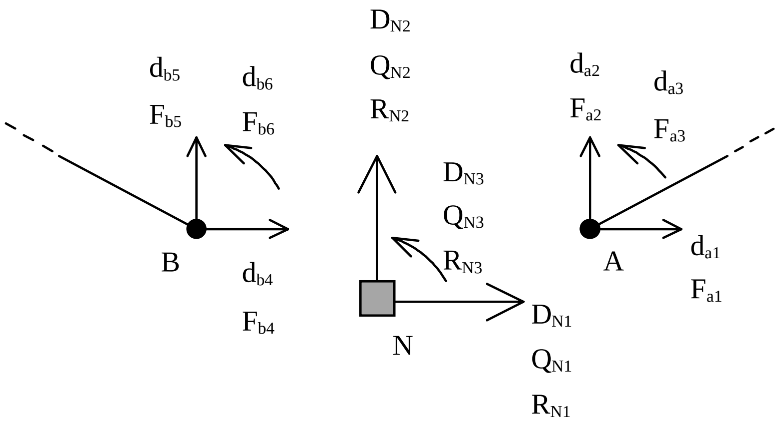

Figure 3: Node connecting several beams.

The number of terms in the left-hand side of (12) depends on the number of beams that are connected to the node. Vector $\underset{\sim}{F}_a{}'$ includes the first three components of $\underset{\sim}{F}_a$, while vector $\underset{\sim}{F}_b{}''$ includes the last three components of $\underset{\sim}{F}_b$. Reaction components ($\underset{\sim}{R}$) are only present in constrained degrees of freedom.

In node *C* the equilibrium equations are

$$F_{a4} + F_{b1} = 0;\ F_{a5} + F_{b2} = 0;\ F_{a6} + F_{b3} = 0 \qquad (13)$$

To enforce displacement compatibility in node *C* (see Figure 2), it is required that

$$d_{b1} = d_{a4};\ d_{b2} = d_{a5} \qquad (14)$$

The plastic rotation θ_C is defined by

$$\theta_C = d_{b3} - d_{a6} \qquad (15)$$

In nodes A and B the conditions (14) and (15) become

$$d_{a1} = D_{N1};\ d_{a2} = D_{N2};\ d_{b4} = D_{N1};\ d_{b5} = D_{N2} \qquad (16)$$

$$\theta_A = d_{a3} - D_{N3};\ \theta_B = D_{N3} - d_{b6} \qquad (17)$$

In equations (16) and (17), $\underset{\sim}{D}_N$ represents the displacements at the node that is adjacent to the appropriate beam extremity.

To guarantee that in node *A* no plastic rotations occur before the plastic moment is reached (see Figures 2 and 3), one of the following three conditions must be true

$$\theta_A = 0 \quad or \quad F_{a3} = M_p \quad or \quad F_{a3} = -M_p \qquad (18)$$

This requirement is equivalent to the following complementary condition

$$\theta_A\left(M_p^2 - F_{a3}^2\right) = 0 \tag{19}$$

Similarly, in nodes *B* and *C* the complementary conditions are

$$\theta_B\left(M_p^2 - F_{b6}^2\right) = 0 \tag{20}$$

$$\theta_C\left(M_p^2 - F_{a6}^2\right) = 0 \tag{21}$$

5 Integrated formulation

When an integrated formulation is adopted to solve an optimization problem, design and behavior variables are simultaneously present in the nonlinear program and, consequently, no sensitivity analysis is required [5]. This growth in the size of the mathematical program is a disadvantage in terms of computer requirements, but facilitates the formulation of certain types of optimization problems. When the size of the problem is not too large, or the available computer resources are sufficient, the gains in versatility compensate the loss in computation time. When the number of design variables is very large and the number of behavior variables is of the same order of magnitude, an integrated formulation solved by a second-order optimization method might be the best approach [4].

The optimization of frames with material nonlinear behavior using an integrated formulation requires the preparation of a mathematical program in the form (1)-(3). In this section, a description of its objective function and constraints is performed.

5.1 Objective function

The objective function is the cost of the structure given by

$$f\,(\underset{\sim}{x}) = \sum_{i=1}^{NB} c_i A_i L_i \tag{22}$$

In this expression, *NB* is the number of bars, *c* is the cost per unit volume of the material used in each beam, *A* is the area of the cross section and *L* is the beam length. When the same material is used in all the bars, the coefficients c_i may be unitary.

5.2 Equality constraints

All the equations referred in Section 4 must be present in the mathematical program as equality constraints. A few more equations must be also included, such as the relationship between the beam length and its subparts

$$L = a + b \tag{23}$$

When a square hollow section is used (see Figure 1), the following equations are required in order to define the cross section area (A), the moment of inertia (I) and the fully plastic moment (M_p).

$$A = 4w(B - w) \tag{24}$$

$$I = 2w\left(B^3 - 3B^2w + 4Bw^2 - 2w^3\right)\Big/3 \tag{25}$$

$$M_p = \sigma w\left(3B^2 - 6Bw + 4w^2\right)\Big/2 \tag{26}$$

To account for the boundary conditions, the following constraints must be included

$$D_j = \overline{D} \tag{27}$$

In equation (27), j loops over the prescribed degrees of freedom and $\overline{D}$ is a predefined value. The following constraint removes the reaction in nonprescribed degrees of freedom.

$$R_k = 0 \tag{28}$$

In a beam with a distributed load, a plastic hinge may be formed close to mid-span in the position of maximum bending moment (see Figure 2). In order to locate the point C in the correct position, the following constraint is required

$$F_{a5} = 0 \tag{29}$$

This constraint locates the point C in the position of null shear force. In this position the bending moment reaches its maximum value in the span. As the moments in A, B and C are limited (see Section 5.3), the moments in all the beam points are guaranteed to be bound by the fully plastic moment (M_p).

To avoid unreal cross sections, some dimensions must be predefined. In the square hollow section used in this work (see Figure 1) the width w was fixed.

5.3 Inequality constraints

Bending moments in the points A, B and C must be bound by the fully plastic moment M_p. This is accomplished by the following inequality constraints

$$-M_p \le F_{a3} \le M_p \tag{30}$$

$$-M_p \le F_{a6} \le M_p \tag{31}$$

$$-M_p \le F_{b6} \le M_p \tag{32}$$

In order to avoid extensive plastic deformations, plastic hinge rotations should be limited. These constraints are directly imposed on the variables θ_A, θ_B and θ_C defined by (15) and (17). The predefined limit depends on the type of material and on the shape of the cross section.

All the independent design variables must be bound using side constraints in order to avoid unreal solutions and to improve the robustness of the iterative process.

6 Rectangular portal frame

To illustrate the proposed formulation a simple example is presented. In Figure 4 a rectangular portal frame with a horizontal point load and a vertical distributed load is shown. Steel and a square hollow section with constant web thickness (1 cm) were used in the frame bars (see Figure 1). In the optimal solution both columns must have the same cross section and plastic hinge rotations cannot exceed 0.01 rad.

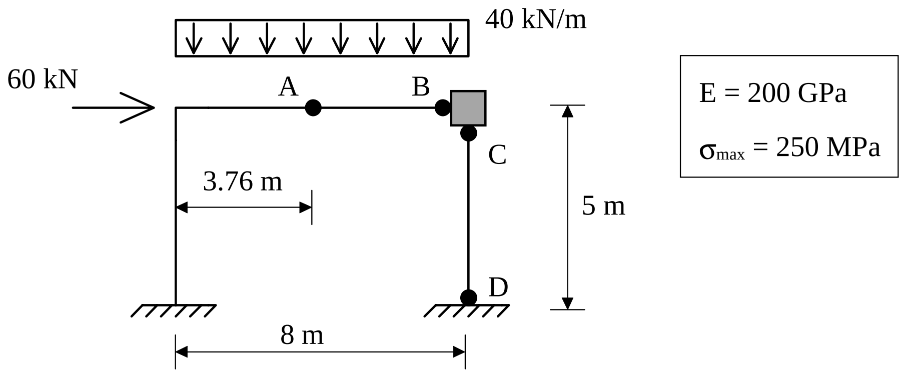

Figure 4: Optimization of a rectangular portal frame - plastic hinge locations.

When the optimized frame is loaded, four plastic hinges are formed at points A, B, C and D (see Figure 4). The load cannot be increased because the maximum plastic rotation has been reached at hinges B and C. Plastic rotation at hinges A and D is less than 0.01 rad. The volume of the optimized frame is 0.157 m^3 and the horizontal displacement at point A is 5.8 cm.

The rectangular portal frame was also optimized considering a linear elastic behavior. With this assumption the optimal volume is 0.175 m^3 (11% higher) and the horizontal displacement of point A is 2.7 cm (approximately half).

7 Conclusions

The optimization of frames with material nonlinear behavior can be based on sensitivity analysis or an integrated formulation. Each approach has its advantages and drawbacks, essentially related to the balance between efficiency and versatility. Increasing performance and storage capacity of low cost computer platforms tends to decrease the concerns related to computer resources and, consequently, makes versatility, robustness and accuracy important characteristics of the optimization algorithm. The combination of a second-order optimization method with an integrated formulation implies some sacrifices in terms of computation time and maximum problem size, but offers the user a way of obtaining very accurate solutions and the ability to easily modify the mathematical program. These characteristics combined with the user-friendliness of the computer program will certainly help to make the use of optimization tools more common in engineering practice.

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